JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **25**, No. 3, August 2012

MULTIPLIERS IN HILBERT ALGEBRAS

Kyung Ho Kim*

Abstract. In this paper, we introduce the concept of multiplier in a Hilbert algebra and obtained some properties of multipliers. Also, we introduce the simple multiplier and characterized the kernel of multipliers in Hilbert algebras.

1. Introduction

The concept of Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implication in intuitionistic and other classical logics. In 60-ties, these algebras were studied especially, by A. Horn and A. Diego [3] from algebraic point of view. Recently, the Hilbert algebras were treated by D. Buseneag [1, 2]. In [4] a partial multiplier on a commutative semigroup $(A, \cdot)$ has been introduced as a function F from a nonvoid subset D_F of A into A such that $F(x) \cdot y = x \cdot F(y)$ for all $x, y \in D_F$. In this paper, we introduce the concept of multiplier in a Hilbert algebra and obtained some properties of multipliers. Also, we introduce the simple multiplier and characterized the kernel of multipliers in Hilbert algebras.

2. Preliminaries

A *Hilbert algebra* is a triple $(H, *, 1)$, where H is a nonempty set, " $*$ " is a binary operation on H, $1 \in H$ is an element such that the following three axioms are satisfied for every $x, y, z \in H$:

(H1) $x * (y * x) = 1$,
(H2) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,

Received December 20, 2011; Accepted June 20, 2012.
2010 Mathematics Subject Classification: Primary 06F35, 03G25, 08A30.
Key words and phrases: Hilbert algebra, commutative, multiplier, simple multiplier.
The research was supported by a grant from the Academic Research Program of Korea National University of Transportation in 2012.

(H3) if $x * y = y * x = 1$ then $x = y$.

If H is a Hilbert algebra, then the relation $x \leq y$ if and only if $x * y = 1$ is a partial order on H, which will be called the *natural ordering* on H. With respect to this ordering, 1 is the largest element of H. A subset S of a Hilbert algebra H is called a *subalgebra* of H if $x * y \in S$ for all $x, y \in S$. A mapping $f : H \to H'$ of Hilbert algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in H$. Also, f is said to be *non-expansive* if $f(x) \leq x$.

In a Hilbert algebra H, the following properties hold for all $x, y, z \in H$,

(H4) $x * x = 1$,
(H5) $x * 1 = 1$,
(H6) $x * (y * z) = (x * y) * (x * z)$,
(H7) $1 * x = x$,
(H8) $x * (y * z) = y * (x * z)$,
(H9) $x * ((x * y) * y)) = 1$,
(H10) $x \leq y$ implies $z * x \leq z * y$ and $y * z \leq x * z$.

For any x, y in a Hilbert algebra H, we define $x \vee y = (y * x) * x$. Note that $x \vee y$ is an upper bound of x and y. A Hilbert algebra H is said to be *commutative* if for all $x, y \in H$,

$$(y * x) * x = (x * y) * y, \quad \text{i.e.,} \quad x \vee y = y \vee x.$$

Let H be a Hilbert algebra. A subset F of Hilbert algebras H is called a *deductive system* if it satisfies

(1) $1 \in F$,
(2) If $x \in F$ and $x * y \in F$, then $y \in F$ for all $x, y \in H$.

3. Multipliers in Hilbert algebras

In what follows, let H denote a Hilbert algebra unless otherwise specified.

Definition 3.1. Let $(H, *, 1)$ be a Hilbert algebra. A self-map f of H is called a *multiplier* if

$$f(x * y) = x * f(y)$$

for all $x, y \in H$.

Example 3.2. Let $H = \{1, a, b, c\}$ be a set in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	c
c	1	1	1	1

It is easy to check that $(H, *, 1)$ is a Hilbert algebra. Define a map $f : H \to H$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, b, c \\ a & \text{if } x = a \end{cases}$$

Then it is easy to check that f is a multiplier of Hilbert algebra H.

Example 3.3. Let $H = \{1, a, b, c\}$ be a set in which "$*$" is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

It is easy to check that $(H, *, 1)$ is a Hilbert algebra. Define a map $f : H \to H$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, a \\ b & \text{if } x = b, c \end{cases}$$

Then it is easy to check that f is a multiplier of a Hilbert algebra H.

Example 3.4. The identity mapping ϵ, the unit mapping $\iota : a \longmapsto 1$ are multipliers of H.

Lemma 3.5. *Let f be a multiplier in a Hilbert algebra H. Then we have $f(1) = 1$.*

Proof. Substituting $f(1)$ for x and 1 for y in Definition 3.1, we obtain $f(1) = f(f(1) * 1) = f(1) * f(1) = 1$. □

Proposition 3.6. *Let f be a multiplier in a Hilbert algebra H. Then*
(1) $x \leq f(x)$ for all $x \in H$,
(2) $f(x) * f(y) \leq f(x * y)$ for all $x, y \in H$.

Proof. (1) Putting $x = y$ in Definition 3.1, we get $1 = f(1) = f(x * x) = x * f(x)$, that is, $x \leq f(x)$.

(2) Since $x \leq f(x)$ for all $x \in H$, it follows from (H10) that $f(x) * f(y) \leq x * f(y) = f(x * y)$ for all $x, y \in H$. □

Let H be a Hilbert algebra and f be a multiplier on H. If $x \leq y$ implies $f(x) \leq f(y)$, f is said to be *isotone*.

Proposition 3.7. *Let f is a multiplier of H and an endomorphism. Then f is isotone.*

Proof. Let $x \leq y$ for any $x, y \in H$. Then $x * y = 1$. Now $f(x) * f(y) = f(x * y) = f(1) = 1$, which implies $f(x) \leq f(y)$. This completes the proof. □

Theorem 3.8. *Let f be a multiplier of H and non-expansive. Then f is an endomorphism of H.*

Proof. Let f be a multiplier of H and non-expansive. Then we have $f(x) \leq x$. Hence $f(x * y) = x * f(y) \leq f(x) * f(y)$ by (H10) and so $f(x) * f(y) = f(x * y)$ from Proposition 3.6 (2) and (H3). This completes the proof. □

The converse of Theorem 3.8 is not true in general.

Example 3.9. In Example 3.3, it is easy to know that f is an homomorphism of H. But $f(a) * a = 1 * a = a$, that is, $f(a) \not\leq a$. Hence the converse of Theorem 3.8 is not true in general.

Let H be a Hilbert algebra and f_1, f_2 two self-maps. We define $f_1 \circ f_2 : H \to H$ by

$$(f_1 \circ f_2)(x) = f_1(f_2(x))$$

for all $x \in H$.

Proposition 3.10. *Let H be a Hilbert algebra and f_1, f_2 two multipliers of H. Then $f_1 \circ f_2$ is also a multiplier of H.*

Proof. Let H be a Hilbert algebra and f_1, f_2 two multipliers of H. Then we have

$$\begin{aligned}(f_1 \circ f_2)(a * b) &= f_1(f_2(a * b)) \\ &= f_1(a * f_2(b)) \\ &= a * f_1(f_2(b)) \\ &= a * (f_1 \circ f_2)(b)\end{aligned}$$

for any $a, b \in H$. This completes the proof. □

Let H be a Hilbert algebra and f_1, f_2 two self-maps. We define $f_1 \vee f_2 : H \to H$ by

$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$$

for all $x \in H$.

Proposition 3.11. *Let H be a Hilbert algebra and f_1, f_2 two multipliers of H. Then $f_1 \vee f_2$ is also a multiplier of H.*

Proof. Let H be a Hilbert algebra and f_1, f_2 two multipliers of H. Then we have

$$\begin{aligned}(f_1 \vee f_2)(a * b) &= f_1(a * b) \vee f_2(a * b) = (a * f_1(b)) \vee (a * f_2(b)) \\ &= ((a * f_2(b)) * (a * f_1(b))) * (a * f_1(b)) \\ &= (a * (f_2(b) * f_1(b))) * (a * f_1(b)) \\ &= a * ((f_2(b) * f_1(b)) * f_1(b)) \\ &= a * (f_1(b) \vee f_2(b)) \\ &= a * (f_1 \vee f_2)(b)\end{aligned}$$

for any $a, b \in H$. This completes the proof. □

Let H_1 and H_2 be two Hilbert algebras. Then $H_1 \times H_2$ is also a Hilbert algebra with respect to the point-wise operation given by

$$(a, b) * (c, d) = (a * c, b * d)$$

for all $a, c \in H_1$ and $b, d \in H_2$.

Proposition 3.12. *Let H_1 and H_2 be two Hilbert algebras. Define a map $f : H_1 \times H_2 \to H_1 \times H_2$ by $f(x, y) = (x, 1)$ for all $(x, y) \in X_1 \times X_2$. Then f is a multiplier of $H_1 \times H_2$ with respect to the point-wise operation.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in H_1 \times H_2$. The we have

$$\begin{aligned}f((x_1, y_1) * (x_2, y_2)) &= f(x_1 * x_2, y_1 * y_2) \\ &= (x_1 * x_2, 1) \\ &= (x_1 * x_2, y_1 * 1) \\ &= (x_1, y_1) * (x_2, 1) \\ &= (x_1, y_1) * f(x_2, y_2).\end{aligned}$$

Therefore f is a multiplier of the direct product $H_1 \times H_2$. □

Denote by $\mathcal{M}(H)$ the set of all multipliers of H.

DEFINITION 3.13. For any $f \in \mathcal{M}(H)$, we define the *kernel* of f as follows:

$$Kerf := \{x \in H \mid f(x) = 1\}.$$

PROPOSITION 3.14. *Let f be a multiplier of H. Then $Kerf$ is a subalgebra of H.*

Proof. Clearly, $1 \in Kerf$ and so $Kerf$ is nonempty. For any $x, y \in Kerf$, we have $f(x * y) = x * f(y) = x * 1 = 1$, and so $x * y \in Kerf$. Hence $Kerf$ is a subalgebra of H. □

THEOREM 3.15. *Let f be a multiplier and an endomorphism of H. Then $Kerf$ is a deductive system of H.*

Proof. Clearly, $1 \in Kerf$ since $f(1) = 1$. Let $x \in Kerf$ and $x * y \in Kerf$. Then $1 = f(x * y) = f(x) * f(y) = 1 * f(y) = f(y)$, and so $y \in Kerf$. This completes the proof. □

PROPOSITION 3.16. *Let $f \in \mathcal{M}(H)$. Then f is one-one if and only if $Kerf = \{1\}$.*

Proof. Necessity is obvious. Assume that $Kerf = \{1\}$. Let $x, y \in H$ be such that $f(x) = f(y)$. Then $f(x) * f(y) = 1$. Since $x \le f(x)$ for all $x \in H$, it follows from (H10) that $1 = f(x) * f(y) \le x * f(y) = f(x * y)$ so that $f(x * y) = 1$, that is, $x * y \in Kerf = \{1\}$. Hence $x * y = 1$. Similarly, we have $y * x = 1$ and so $x = y$. This completes the proof. □

PROPOSITION 3.17. *Let f be an isotone multiplier of H. If $x \le y$ and $x \in Kerf$ for any $y \in H$, then $y \in Kerf$.*

Proof. Let f be an isotone multiplier. If $x \le y$ and $x \in Kerf$, then $f(x) \le f(y)$, and so $1 = f(x) * f(y) = 1 * f(y) = f(y)$. Hence $y \in Kerf$. □

PROPOSITION 3.18. *Let f be a multiplier of H. If f is isotone and $f^2 = f$, $Kerf$ is a deductive system of H.*

Proof. Let f be isotone and $x, x * y \in Kerf$, respectively. Then we have $f(x) = 1$ and $f(x * y) = 1$. Thus we have $1 = f(x * y) = x * f(y)$, which implies $x \le f(y)$. Since f is isotone, we get $1 = f(x) \le f(f(y)) = f(y)$. This implies $f(y) = 1$, that is, $y \in Kerf$. This completes the proof. □

Let H be a Hilbert algebra and f be a multiplier of H. Define a set $Fix_f(H)$ by

$$Fix_f(H) := \{x \in H \mid f(x) = x\}.$$

Proposition 3.19. *Let f be a multiplier in a Hilbert algebra H. Then $Fix_f(H)$ is a subalgebra of H.*

Proof. Let $x, y \in Fix_f(H)$. Then we have $f(x) = x$ and $f(y) = y$. Hence $f(x * y) = x * f(y) = x * y$, and so $x * y \in Fix_f(H)$. This proves that $Fix_f(H)$ is a subalgebra of H. □

Proposition 3.20. *Let H be a Hilbert algebra and f a multiplier of H. If $x \in Fix_f(H)$, then $x \vee y \in Fix_f(H)$.*

Proof. Let $x \in Fix_f(H)$. Then we have $f(x) = x$, and so

$$\begin{aligned} f(x \vee y) &= f(y * x) * x) \\ &= (y * x) * f(x) \\ &= ((y * x) * x = x \vee y. \end{aligned}$$

This completes the proof. □

Proposition 3.21. *Let H be a commutative Hilbert algebra and f a multiplier of H. If $x \in Fix_f(H)$ and $x \leq y$, then $y \in Fix_f(H)$.*

Proof. Let H be a commutative Hilbert algebra. If $x \in Fix_f(H)$ and $x \leq y$, then we have

$$\begin{aligned} f(y) &= f(1 * y) = f((x * y) * y) \\ &= f((y * x) * x) = (y * x) * f(x) \\ &= (y * x) * x = (x * y) * y \\ &= 1 * y = y. \end{aligned}$$

This completes the proof. □

Proposition 3.22. *For any $p \in H$, the mapping $\alpha_p(a) = p * a$ is a multiplier of H.*

Proof. Let $p \in H$. Then we have

$$\alpha_p(a * b) = p * (a * b) = a * (p * b) = a * \alpha_p(b).$$

This completes the proof. □

Proposition 3.23. *For any $p \in H$, the mapping $\beta_p(a) = p * (p * a)$ is a multiplier of H.*

Proof. Let $p \in H$. Then we have

$$\begin{aligned} \beta_p(a * b) &= p * (p * (a * b)) \\ &= p * (a * (p * b)) \\ &= a * (p * (p * b)) \\ &= a * \beta_p(b) \end{aligned}$$

for all $a, b \in H$. This completes the proof. $\square$

PROPOSITION 3.24. *For any $p \in H$, the multiplier $\beta_p(a) = p * (p * a)$ is a homomorphism of H.*

Proof. Let $p \in H$. Then we have

$$\begin{aligned}\beta_p(a * b) &= p * (p * (a * b))\\ &= p * ((p * a) * (p * b))\\ &= (p * (p * a)) * (p * (p * b))\\ &= \beta_p(a) * \beta_p(b)\end{aligned}$$

for all $a, b \in H$. This completes the proof. $\square$

PROPOSITION 3.25. *Let H be a Hilbert algebra. If $a \leq b$ for any $a, b \in H$, we have $\beta_p(a * b) = 1$.*

Proof. Let $a \leq b$. Then $a * b = 1$. Thus we have $\beta_p(a * b) = \beta_p(1) = p * (p * 1) = p * 1 = 1$. This completes the proof. $\square$

We call the multiplier $\alpha_p(a) = p * a$ of Example 3.22 as *simple multiplier.*

PROPOSITION 3.26. *For every $p \in H$, the simple multiplier α_p of H is an endomorphism of H.*

Proof. Let $a, b \in H$. Using (H6), we have

$$\alpha_p(a * b) = p * (a * b) = (p * a) * (p * b) = \alpha_p(a) * \alpha_p(b).$$

Hence α_p is an endomorphism of H. $\square$

PROPOSITION 3.27. *The simple multiplier α_1 is an identity function of H.*

Proof. For every $a \in H, \alpha_1(a) = 1 * a = a$. This completes the proof. $\square$

PROPOSITION 3.28. *Let H be a Hilbert algebra. Then, for each $p \in H$, we have $\alpha_p(x \vee p) = 1$.*

Proof. For each $p \in H$, we have

$$\begin{aligned}\alpha_p(x \vee p) = \alpha_p((p * x) * x) &= p * ((p * x) * x)\\ &= (p * x) * (p * x) = 1,\end{aligned}$$

for any $x \in H$. This completes the proof. $\square$

References

[1] D. Busneag, *A note on deductive systems of a Hilbert algebras,* Kobe J. Math **2** (1985), 29-35.

[2] D. Busneag, *Hilbert algebras of fractions and maximal Hilbert algebras of quotients,* Kobe J. Math **5** (1988), 161-172.

[3] A. Diego, *Sur les algébras de Hilbert,* Ed. Hermann, Colléction de Logique Math. Serie A **21** (1999), 1-52.

[4] R. Larsen, *An Introduction to the Theory of Multipliers*, Berlin, Springer-Verlag, 1971.

*
Department of Mathematics,
Korea National University of Transportation
Chungju 380-702, Republic of Korea
E-mail: ghkim@ut.ac.kr